Example- Bernoulli

Toss a biased coin once which shows a head with probability p.

Define random variable X as

$$X = \begin{cases} 1 & \textit{for a head} \\ 0 & \textit{for a tail} \end{cases}$$

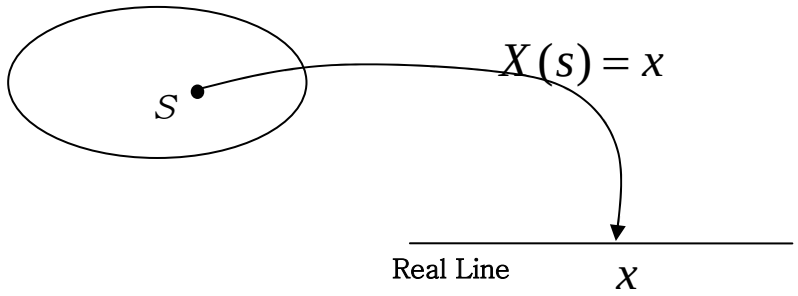
- Range: $S_X = \{0,1\}$
- $\Pr[X = 1] = \Pr[\text{a head}] = p$
- $\Pr[X = 0.5] = 0,\ \Pr[X \geq 0.1] = 0.5,\ \Pr[X > 0] = 0.5,\ \Pr[X \geq 2] = 0,\ \Pr[X > -3] = 1$

X is called a Bernoulli random variable.

X is a discrete random variable.

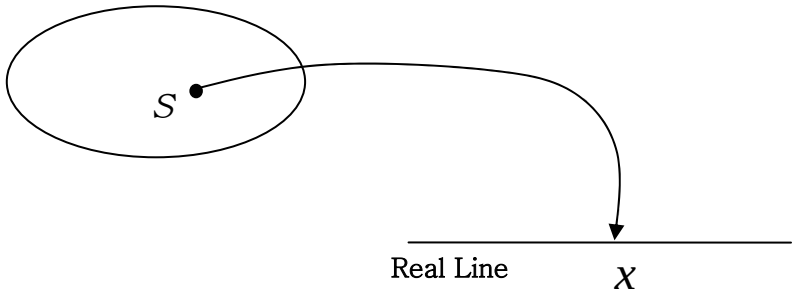
Random Variable

Definition



- Random Variable X is a mapping that maps each outcome s in the sample space to a unique real number x, $-\infty < x < \infty$.
- You must specify how the mapping X works.
  ” Let X be the random variable indicating the number of packets waiting ….”
  ” X is a binary random variable. $X=1$ if the outcome satisfies …”
- Sometimes the outcome itself is a real number. In that case, X is simply the outcome.

Mapping – Definitions



- Range (of the random variable X) $S_X = \{x : x = X(s) \ \textit{for some}\ s \in S\}$
- In general, X is a many-to-one mapping.
- Inverse Operation $X^{-1}(x)$: for any $x \in S_X$, $X^{-1}(x) = \{s : X(s) = x\}$
- Inverse Image $X^{-1}(J)$: for any $J \subset S_X$, $X^{-1}(J) = \{s : X(s) \in J\}$
- We make use of $\Pr[X \in J] = \Pr[\{s : s \in X^{-1}(J)\}]$

Example - Geometric

Toss a biased coin until we see the first head.

X is the number of times we toss the coin until we see the first head.

- Range: $S_X = \{1, 2, 3, \cdots\}$
- $\Pr[X = j] = (1-p)^{j-1} p \quad for \;\; j = 1, 2, 3, \cdots,$ where p is the probability of a head

X is a called a geometric random variable.

X is a discrete random variable.

Correct English: *heads* is an adjective.

To be correct grammatically, p is the probability the coins falls down heads.

Example – Exponential Random Variable

Measure the time between any arbitrary pair of successive packet arrivals at a router.

Let X denote the inter-arrival time.

- Range: $S_X = \{0 < r < \infty\}$
- Experiments show $\Pr[X \le x] = 1 - e^{-\lambda x}$ for $x \ge 0,$ where λ is the packet arrival rate.
- $\Pr[X = 5] = 0$
- $\Pr[0 < X < 5] \equiv \Pr[0 \le X \le 5] = 1 - e^{-5\lambda}$

X is called an exponential random variable.

X is a continuous random variable.

Types of Random Variables

- Discrete RV
- Continuous RV

Discrete Random Variable

A range has a finite number of points $\{x_1, x_2, \cdots, x_K\}$,

or is countably infinite $\{x_1, x_2, \cdots, x_K, \cdots\}$

Notation

In the notation $\Pr[X = x]$ or $\Pr\{X = x\}$,

X is a random variable, and x is a constant value.

Cumulative Distribution Function of a random variable

Definition:

$$F_X(x) = P[X \le x]$$
$$= P[\{s : -\infty \le X(s) \le x\}]$$

Other terms are cdf and Probability Distribution Function (PDF).

cdf Properties:

① $0 \le F_X(x) \le 1$

② $F_X(\infty) = 1$

③ $F_X(-\infty) = 0$

④ $F_X(x)$ is a non-decreasing function of x

⑤ $F_X(x)$ is continuous from the right: that is, $F_X(b) = \lim_{h \to 0} F_X(b+h)$

Example - Uniform

Throw dart at a spinning wheel. X is the phase where the dart hits the wheel.

Find the cdf of X.

First we note $S_X = \{0 \le x \le 2\pi\}$.

Within the range, $P[x - \Delta/2 \le X \le x + \Delta/2] = \Delta/2\pi$ for any phase x

Therefore,

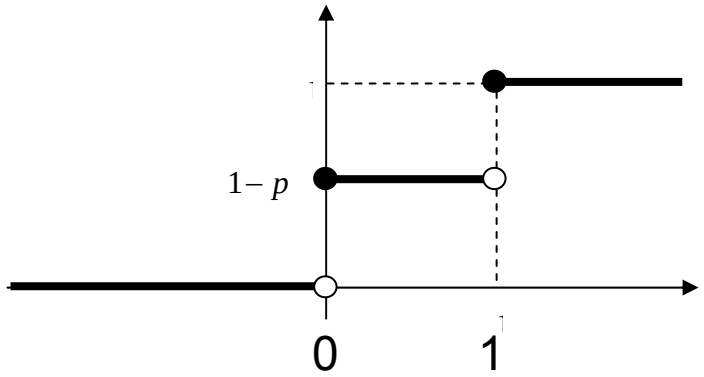
$$F_X(x) = \Pr\{X \le x\} = \begin{cases} 1 & x \ge 2\pi \\ \dfrac{x}{2\pi} & 0 \le x < 2\pi \\ 0 & x < 0 \end{cases}$$

Example - Bernoulli

Let X be a Bernoulli random variable.

$$X = \begin{cases} 1 & \textit{for a head} \\ 0 & \textit{for a tail} \end{cases}, \quad \text{with prob } p$$

$$F_X(x) = \Pr\{X \le x\} = \begin{cases} 1 & x \ge 1 \\ 1-p & 0 \le x < 1 \\ 0 & x < 0 \end{cases}$$



Discrete RVs have discontinuities in their cdf's.

The values of their cdf's are taken approaching from the right.

Probability density function (pdf)

Definition:

$$f_X(x) \triangleq \lim_{\Delta \to 0} \frac{F_X(x+\Delta) - F_X(x)}{\Delta} = \frac{d}{dx} F_X(x)$$

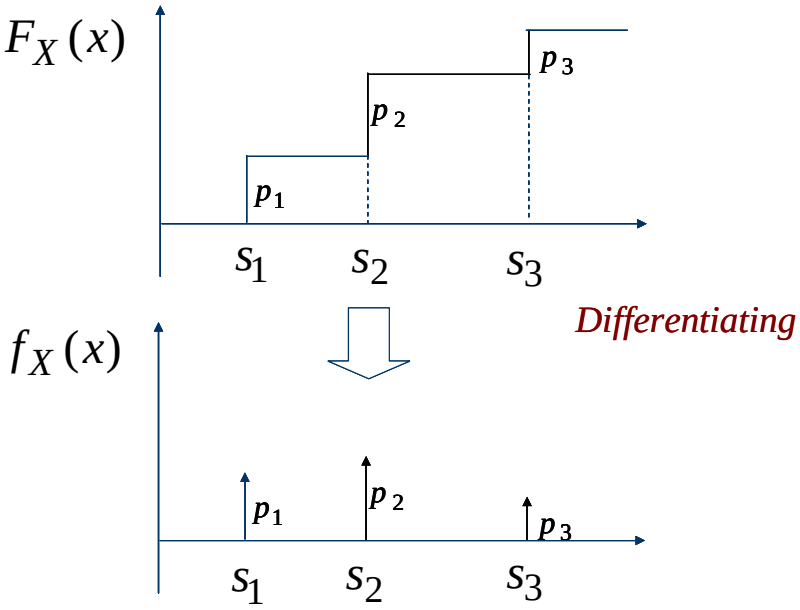
- Differentiate the cdf to get the pdf
- A pdf is the rate at which its cdf increases
- Integrate pdf to get its cdf: $F_X(x) = \int_{-\infty}^{x} f_X(u)du$
- Finding the probability of interest: $\Pr\{a \le X \le b\} = F_X(b) - F_X(a) = \int_{a}^{b} f_X(u)du$

Guideline to finding discrete distributions

- The pdf of a discrete random variable is more commonly called the ***probability mass function*** (pmf). When the range is $\{s_0, s_1, s_2, \cdots\}$, $p_k \triangleq \Pr\{X = s_k\}$.

- For a discrete random variable, find pmf first and then cdf $F_X(x) = \Pr\{X \le x\} = \sum_{s_i \le x} p_i$

- You can also differentiate cdf of a discrete rv. You get a delta function pdf.



Guideline to finding continuous distributions

- For a continuous random variable, find the cdf first and then its pdf from
  $f_X(x) = \frac{d}{dx} F_X(x)$
- It is wrong to say $\Pr[X = x] = f_X(x)$.
  Always $\Pr[X = x] = 0$.

However $\Pr[a \le X \le b] = \int_a^b f_X(x)\,dx$

Therefore it is true that for a small Δx, $\Pr[x \le X \le x + \Delta x] \cong f_X(x)\Delta x$.

Popular Discrete Distributions

Bernoulli

Probability of seeing a head with a coin with p as the probability of a head:

$$p_1 = \Pr[X = 1] = p,$$

$$p_0 = \Pr[X = 0] = 1 - p$$

Geometric distribution

Probability of seeing the first head at the k-th toss of a coin with p as the probability of a head:

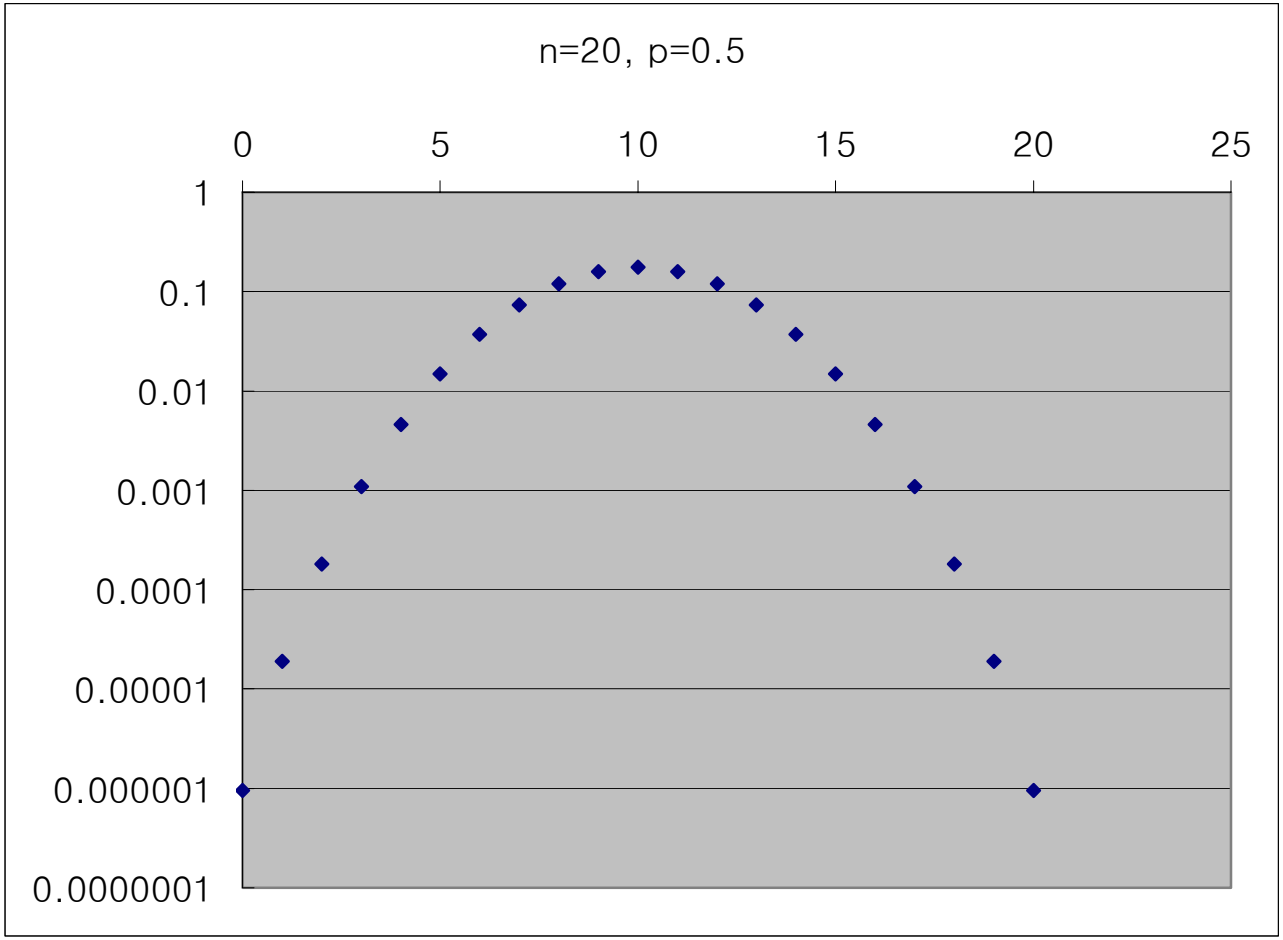
$$p_k = P[X = k] = (1-p)^{k-1} p \ , \ k = 1, 2, 3, \ldots$$

Assume $0 < p < 1$.

Binomial

Probability of seeing k heads in n tosses:

$$p_k = P[X = k] = \binom{n}{k} p^k (1-p)^{n-k} \qquad k = 0,1,2,,...,n$$



Negative Binomial

Probability of seeing the *r*-th head at the *k*-th turn:

$$p_k = \binom{k-1}{r-1} p^{r-1}(1-p)^{k-r}\, p \qquad k = r, r+1, \ldots$$

Poisson

Probability of *k* arrivals during a time interval *t* in a Poisson arrival process with rate λ

$$p_k = \frac{(\lambda t)^k}{k!} e^{-\lambda t} \qquad k = 0, 1, 2, \cdots$$

For all discrete rvs, $\sum_k p_k = 1$

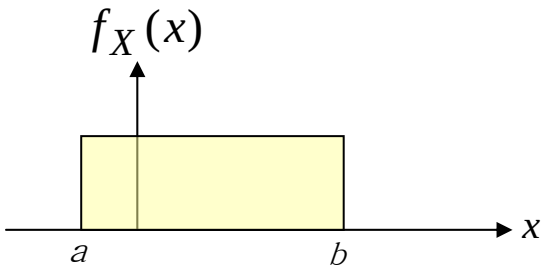
Popular Continuous Distributions

Uniform $U(a,b)$

models a random point within a finite interval.

pdf: $f_X(x) = \dfrac{1}{b-a} \qquad a \le x \le b,\ a < b$

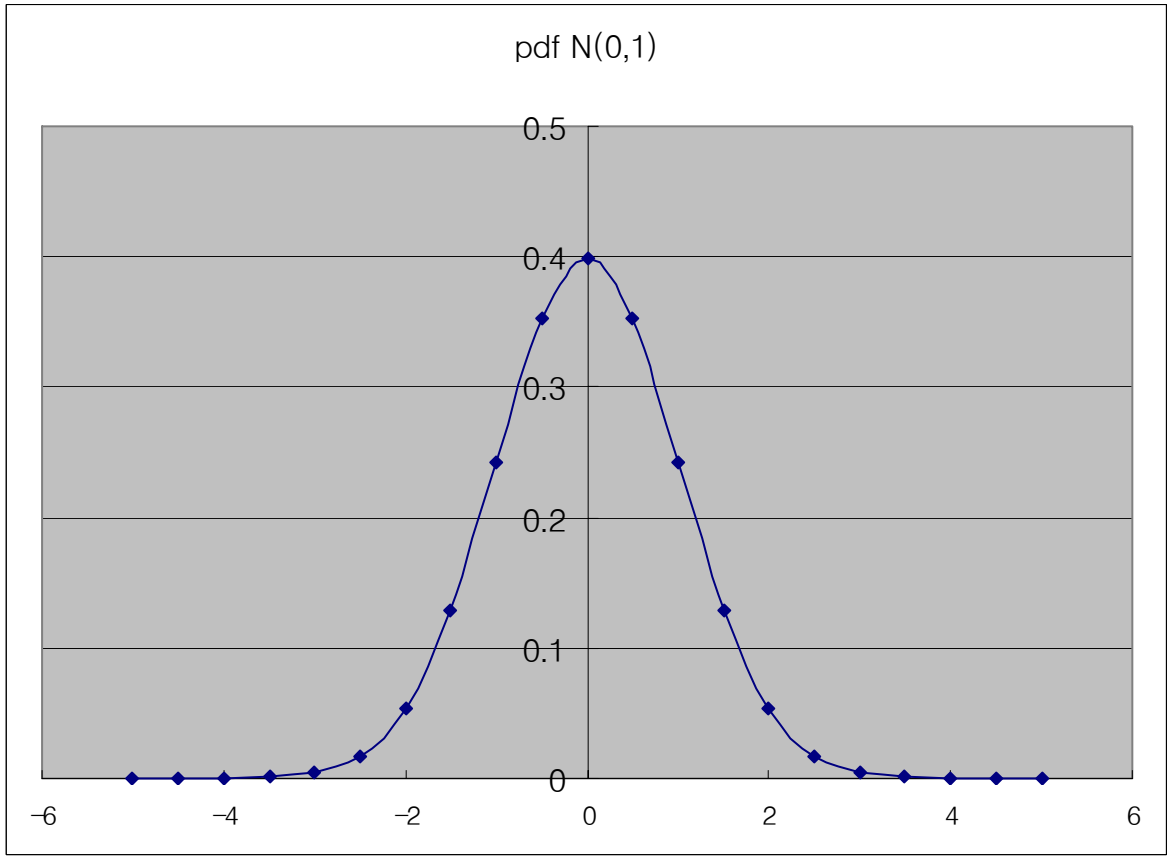
The total area under the pdf should always be 1.



Gaussian $N\left(m,\sigma^2\right)$

models a sum of many independent random variables.

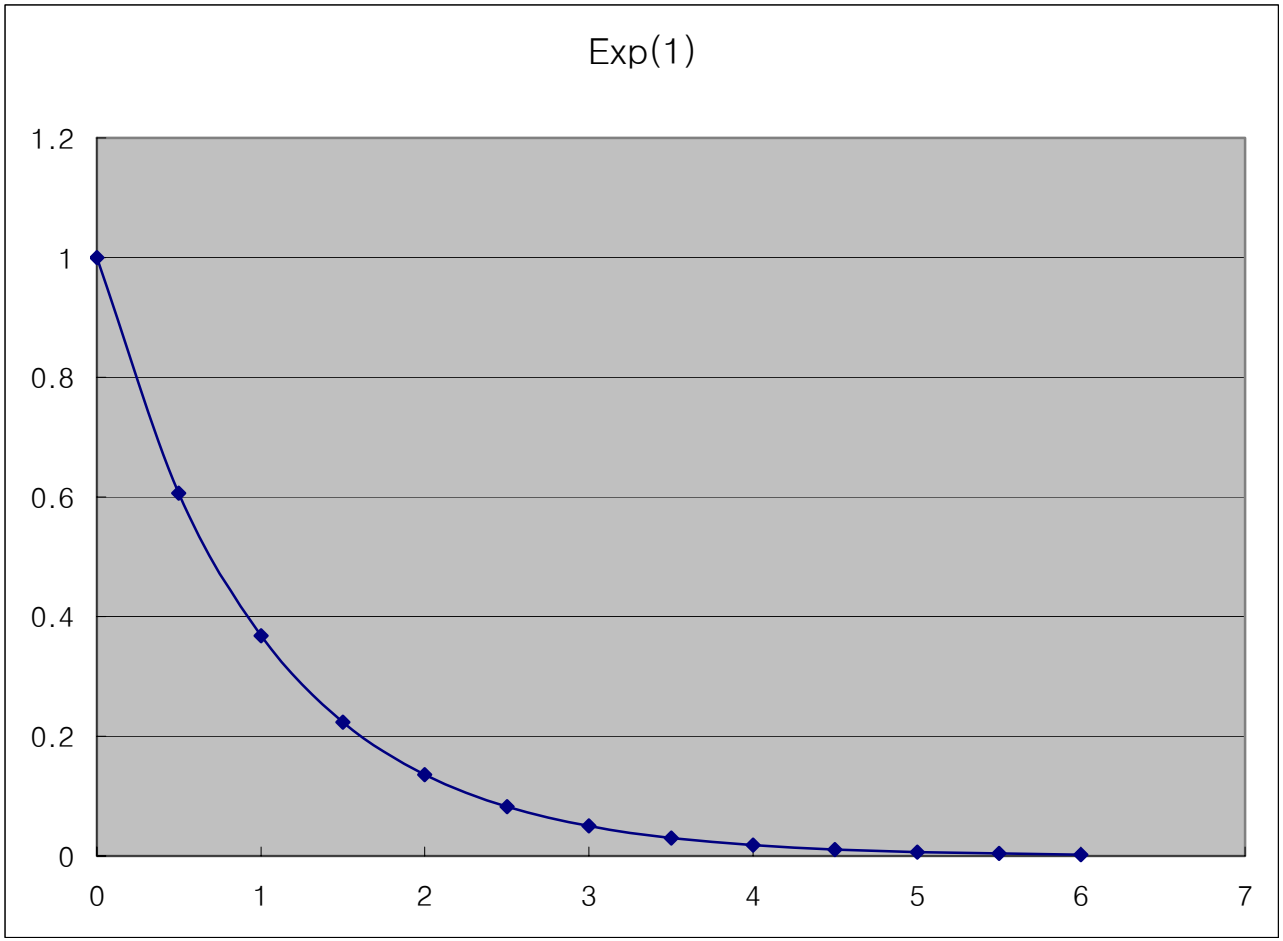
pdf: $f_X(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-m)^2}{2\sigma^2}} \qquad -\infty<x<\infty$



Exponential $Exp(\lambda)$

models the inter-arrival time in a Poisson process with arrival rate λ.

pdf: $f_X(x) = \lambda e^{-\lambda x} \qquad x \geq 0$



m-Erlang $Erl(\lambda, m)$

models the *m*-th arrival time of a poisson process with arrival rate λ.

pdf: $f_X(x) = \dfrac{\lambda(\lambda x)^{m-1}}{(m-1)!} e^{-\lambda x} \qquad x \geq 0,\ \lambda > 0$

Gamma $G(\lambda,\alpha)$

is a general case of m-erlang.

pdf: $f_X(x) = \dfrac{\lambda(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda x}$ $\quad x \geq 0$ with $\alpha > 0,\ \lambda > 0$

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

$$\Gamma(1) = 1$$

$$\Gamma(z+1) = z\Gamma(z) = z!$$

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$$

Raleigh

models the amplitude of the sum of two orthogonal, independent gaussian RVs

pdf: $f_X(x) = \frac{x}{b} e^{-\frac{x^2}{2b}} \qquad x \geq 0 \quad \text{with } b > 0$

Chi-square $\chi(k)$

models the sum of squares of k independent gaussian RVs.

pdf: $f_X(x) = \frac{\frac{1}{2}(\frac{x}{2})^{\frac{k}{2}-1}}{\Gamma(\frac{k}{2})} e^{-\frac{x}{2}} \qquad x \geq 0 \text{ with } \alpha > 0$

is a special case of Gamma: $\alpha = \frac{k}{2}$, *and* $\lambda = \frac{1}{2}$.

Cauchy

models the ratio of two independent Gaussian RV

pdf: $f_X(x) = \dfrac{\alpha/\pi}{x^2 + \alpha^2} \qquad -\infty < x < \infty \text{ with } \alpha > 0$

Laplacian

is a two-sided exponential

pdf: $f_X(x) = \dfrac{\alpha}{2} e^{-\alpha|x|} \qquad -\infty < x < \infty \text{ with } \alpha > 0$