Random Process

Classification of Random Processes

$X(t,\zeta)$

t : time

ζ : outcome in the sample space

t	ζ	$X(t,\zeta)$
Fixed	Fixed	constant
Fixed	Variable	Random variable
Variable	Fixed	waveform
Variable	Variable	random process

- State space: what values can the process take on at anyone time?
- Time index: when can transitions occur?
- Statistical relationship among random samples

State Space

The sample at time t may be a discrete or continuous random variable:

- discrete: number of arrivals, queue length
- continuous: position of an aircraft, speech signal amplitude

[note] Quantization changes a continuous to a discrete random variable.

Time Index

- Discrete: $X_0, X_1, X_2, \cdots$

  Examples) Periodic samples of a speech signal or
  the queue length of a transmit packet buffer.
  Repeated experiment: coin tossing, black jack wins.

- Continuous: $X(t)$

  Transition can occur at any time.
  But we can take time samples: $X_{t_1}, X_{t_2}, X_{t_3}, \cdots$ *or* $X(t_1), X(t_2), X(t_3), \cdots$

Statistical Relationship among Random Samples

In general, a random process is characterized by the joint cdf

$$F_{X_{t_1},X_{t_2},\cdots,X_{t_k}}(x_1,x_2,\cdots,x_k)=\Pr\{X_{t_1}\le x_1,X_{t_2}\le x_2,\cdots,X_{t_k}\le x_k\}$$

for any choice of $t_1<t_2<\cdots<t_k$ and for any k.

However we need a less complicated model that is

- close to the real world
- tractable to analysis.

Frequently used assumptions are:

- stationary and wide-sense stationary random process for analyzing signals
- markov process for analyzing queues

Classes of Random Processes

Stationary random processes

time-shift-invariant:

$$F_{X_{t_1},X_{t_2},\cdots,X_{t_k}}(x_1,x_2,\cdots,x_k) = F_{X_{t_1+\tau},X_{t_2+\tau},\cdots,X_{t_k+\tau}}(x_1,x_2,\cdots,x_k)$$

for any τ, *any choice of* $t_1 < t_2 < \cdots < t_k$, *and for any* k

Wide-sense stationary random processes

Mean: $\overline{X(t)} = m$ for all t

Autocorrelation: $R_X(t_1,t_2) = \overline{X_{t_1} X_{t_2}} = R_X(\tau), \quad \tau = |t_2 - t_1|$ for any t_1 and t_2

Markov processes

Memoryless:

$$F_{X_{t_{k+1}}|X_{t_k},X_{t_{k-1}},\cdots,X_{t_1}}(x_{k+1} \mid x_k, x_{k-1},\cdots,x_1) = F_{X_{t_{k+1}}|X_{t_k}}(x_{k+1} \mid x_k)$$

for any k and for any choice of $t_1 < t_2 < \cdots < t_k < t_{k+1}$

Some motives to study random processes

- Estimation: Observe $X(t)$ and predict $X(t+\tau)$
- Steady state: mean queue size $\overline{N} = \lim_{t\to\infty} \overline{N(t)}$
- Spectral Analysis: $S_X(f)$ power density at frequency *f* of a random process *X(t)*

Moments of a Random Process

Mean

$$m_X(t) = \overline{X(t)}$$

Variance

$$\sigma_X^2(t) = \overline{\{X(t) - m_X(t)\}^2}$$

Autocorrelation

$$R_X(t_1, t_2) = \overline{X_{t_1} X_{t_2}}$$

Autocovariance

$$C_X(t_1, t_2) = \overline{\{X_{t_1} - m_X(t_1)\}\{X_{t_2} - m_X(t_2)\}} = R_X(t_1, t_2) - m_X(t_1) m_X(t_2)$$

Examples of Random Processes

Random phase signal

Suppose $X(t) = \cos(wt + \Theta)$, where Θ is uniform over $(0, 2\pi)$.

Then

$$m_X(t) = \frac{1}{2\pi}\int_0^{2\pi} \cos(wt + \theta)d\theta = 0 \qquad for \quad all \quad t.$$

$$C_X(t_1, t_2) = R_X(t_1, t_2)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos(wt_1 + \theta)\cos(wt_2 + \theta) \quad d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos(wt_1 + wt_2 + 2\theta) + \cos(wt_1 - wt_2)}{2} \quad d\theta$$

$$= 0 + \frac{1}{2}\cos\{w(t_2 - t_1)\} = \frac{1}{2}\cos w\tau$$

is a function of $\tau = t_2 - t_1$.

Note. $\cos A\cos B = \dfrac{\cos(A+B) + \cos(A-B)}{2}$, $\cos(A+B) = \cos A\cos B - \sin A\sin B$

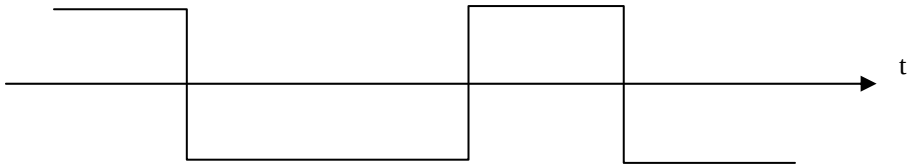
Thus the random phase signal is a wide-sense stationary random process.

Random Telegraph Signal

Signal level is either +1 or –1.

The time between transitions is exponentially distributed with mean $1/\alpha$.

Assume $X(0) = \pm 1$ with an probability 0.5.



❍ We claim $\Pr\{X(t)=1\} = \Pr\{X(t)=-1\} = 0.5$ for any t

$$\Pr\{X(t)=1\} = \Pr\{X(t)=1 \mid X(0)=-1\}\Pr\{X(0)=-1\} + \Pr\{X(t)=1 \mid X(0)=1\}\Pr\{X(0)=1\}$$

$$= \Pr\{\text{an odd number of transitions during } t\}\frac{1}{2} + \Pr\{\text{an even ....}\}\frac{1}{2}$$

$$= \Pr\{\text{any number of transitions during } t\}\frac{1}{2} = \frac{1}{2}$$

By the way,

$$\Pr\{\text{an even number of transitions during } t\}$$

$$= \sum_{k=0}^{\infty} P_{2k}(t) = \sum_{k=0}^{\infty} \frac{(\alpha t)^{2k} e^{-\alpha t}}{(2k)!} = e^{-\alpha t} \sum_{k=0}^{\infty} \frac{(\alpha t)^{2k}}{(2k)!} = e^{-\alpha t} \frac{e^{\alpha t} + e^{-\alpha t}}{2} = \frac{1+e^{-2\alpha t}}{2}$$

$$\Pr\{\text{an odd number of transitions during } t\} = e^{-\alpha t} \frac{e^{\alpha t} - e^{-\alpha t}}{2} = \frac{1-e^{-2\alpha t}}{2}$$

❍ We claim $m_X(t) = 0$ *and* $\sigma_X(t) = 1$ *for all* t

❍ We claim $C_X(t_1,t_2) = R_X(t_1,t_2) = e^{-2\alpha|\tau|}$ *where* $\tau = t_2 - t_1$

$$R_X(t_1,t_2) = \overline{X(t_1)X(t_2)} = (+1)\cdot \Pr\{X(t_2) = X(t_1)\} + (-1)\cdot \Pr\{X(t_2) \neq X(t_1)\}$$

$$= \Pr\{\text{an even number of transitions}\} - \Pr\{\text{an odd}\}$$

$$= \frac{1+e^{-2\alpha|t_2-t_1|}}{2} - \frac{1-e^{-2\alpha|t_2-t_1|}}{2}$$

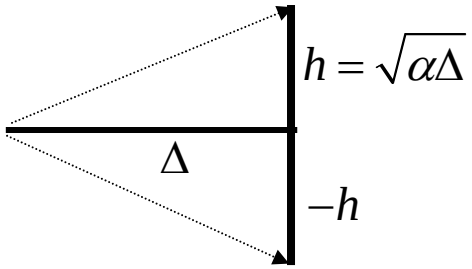
$$= e^{-2\alpha|t_2-t_1|} = e^{-2\alpha|\tau|} \quad \text{where } \tau = t_2 - t_1$$

Thus the random telegraph signal is a wide-sense stationary random process.

Wiener Process and Brownian Motion

Initial condition: $X(0)=0$.

The process makes a transition every Δ, up or down with equal probabilities.



- It can be shown that $m_X(t)=0$ for all t, and $\sigma_X(t)=\alpha t$.
  Model the successive transitions as *iid* random variables $D_1, D_2, D_3, \cdots$.

  Then $\overline{D_j}=0$ *and* $\sigma_{D_j}^2=h^2=\alpha\Delta$.

  How many transitions occur *during* $(0,t)$? *Ans.* $\left\lfloor \frac{t}{\Delta} \right\rfloor$.

  Therefore $X(t)=D_1+D_2+\cdots+D_{\left\lfloor \frac{t}{\Delta} \right\rfloor}$, and $m_X(t)=0$, $\sigma_X(t)=\alpha\Delta\left\lfloor \frac{t}{\Delta} \right\rfloor \cong \alpha t$

  Conclusion: *As* $\Delta\to 0$, $X(t)$ *becomes a gaussian,* $N(0,\alpha t)$ by the central limit theorem.

❍ It can be shown that $C_X(t_1,t_2) = R_X(t_1,t_2) = \alpha \quad \min(t_1,t_2)$

Suppose $t_1 < t_2$.

Then

$$R_X(t_1,t_2) = \overline{X(t_1)X(t_2)} = \overline{\left(D_1 + D_2 + \cdots + D_{\left\lfloor \frac{t_1}{\Delta} \right\rfloor}\right)\left(D_1 + D_2 + \cdots + D_{\left\lfloor \frac{t_2}{\Delta} \right\rfloor}\right)}$$

$$= \overline{D_1^{\,2} + D_2^{\,2} + \cdots + D_{\left\lfloor \frac{t_1}{\Delta} \right\rfloor}^{\,2}} \qquad \text{since} \quad \overline{D_i D_j} = 0 \quad \textit{for} \quad i \neq j$$

$$= \alpha\Delta \quad \left\lfloor \frac{t_1}{\Delta} \right\rfloor \cong \alpha t_1 = \alpha \cdot \min(t_1,t_2)$$

The Wiener process is not a wide-sense stationary random process.